Exercice 1 :

La vraisemblance :

$$\mathcal{L}(Y_{1,1}, Y_{2,1}, \ldots, Y_{n,1}), \ldots) =$$

$$\prod_{i=0}^{1} \prod_{j=1}^{n_i} \left[\frac{(L_{ij}\lambda_i)^{y_{ij}} \, e^{-L_{ij}\lambda_i}}{y_{ij}!} \right]$$

$$\ln \mathcal{L} = \sum_{i=0}^{1} \left[\sum_{j=1}^{n_i} y_{ij} \ln L_{ij} \left(\sum_{j=1}^{n_i} y_{ij}\right) \ln \lambda_i - \lambda_i \sum_{j=1}^{n_i} L_{ij} - \sum_{j=1}^{n_i} y_{ij}! \right]$$

Soit pour $i = 0, 1$

$$\frac{\partial \ln L}{\partial \lambda_i} = \frac{\sum_{j=1}^{n} y_{ij}}{\lambda_i} - \sum_{j=1}^{n_i} L_{ij} = 0$$

$$\hat{\lambda}_i = \frac{\sum_{j=1}^{n_i} Y_{ij}}{\sum_{j=1}^{n_i} L_{ij}} \quad \text{MLE de } \lambda_i .$$

Aussi pour $i = 0, 1$,

$$\frac{\partial^2}{\partial \lambda_i^2} \mathcal{L} = - \frac{\sum_{j=1}^{n_i} y_{ij}}{\lambda_i^2}$$

$E(Y_{ij}) = L_{ij}\lambda_i$ et $\frac{\partial^2}{\partial\lambda_1 \partial\lambda_0} \ln L = 0$, nous avons

$$\mathrm{Var}(\hat{\lambda}_i) = \frac{1}{E\left[-\frac{\partial^2 \ln L}{\partial \lambda_i^2}\right]} = \frac{\lambda_i}{\sum_{j=1}^{n_i} L_{ij}}$$

$$\mathrm{Var}(\ln \hat{\varphi}) = \mathrm{Var}(\ln \hat{\lambda}_1) + \mathrm{Var}(\ln \hat{\lambda}_0)$$

$$\simeq \left(\frac{1}{\lambda_1}\right)^2 \mathrm{Var}(\hat{\lambda}_1) + \left(\frac{1}{\lambda_0}\right)^2 \mathrm{Var}(\hat{\lambda}_0)$$

$$= \frac{1}{\lambda_1 \sum_{j=1}^{n_1} L_{1j}} + \frac{1}{\lambda_0 \sum_{j=1}^{n_0} L_{0j}}$$

$$\frac{\ln \hat{\varphi} - \ln \varphi}{\sqrt{\mathrm{Var}(\ln \hat{\varphi})}} = \frac{\ln \hat{\varphi} - \ln \varphi}{\left(\frac{1}{\sum_{j=1}^{n_1} y_{1j}} + \frac{1}{\sum_{j=1}^{n_0} y_{0j}}\right)^{1/2}} \rightsquigarrow \mathcal{N}(0,1)$$

<u>Quand n est grand</u> :

$$\sqrt{n_1}\,(\hat{\lambda}_1 - \lambda_0) \rightsquigarrow \mathcal{N}(0, I^{-1}(\lambda_1))$$

$$\hat{\lambda}_1 - \lambda_0 \rightsquigarrow \mathcal{N}\left(0, \frac{1}{n I(\lambda_1)}\right)$$

$$\hat{\lambda}_1 \rightsquigarrow \mathcal{N}\left(\lambda_1, \frac{1}{n_1 I(\lambda_1)}\right) \text{ et } \hat{\lambda}_0 \rightsquigarrow \mathcal{N}\left(\lambda_0, \frac{1}{n_0 I(\lambda_0)}\right)$$

$$\frac{\ln\hat{\varphi}-\ln\varphi}{\sqrt{Var(\ln\hat{\varphi})}} = \frac{\ln\hat{\varphi}-\ln\varphi}{\left(\frac{1}{\sum_{j=1}^{n_1} y_{1j}}+\frac{1}{\sum_{j=1}^{n_0} y_{0j}}\right)^{1/2}} \rightsquigarrow N(0,1)$$

CI for φ :

$$\ln\hat{\varphi} \pm z_{1-\frac{\alpha}{2}}\sqrt{Var(\ln\hat{\varphi})}$$

$$= (\ln\hat{\lambda}_1 - \ln\hat{\lambda}_0) \pm z_{1-\frac{\alpha}{2}}\left(\frac{1}{\sum_{j=1}^{n_1} y_{1j}+\sum_{j=1}^{n_0} y_{0j}}\right)^{1/2}.$$

donc IC pour φ :

$$(\hat{\varphi})\exp\left[\pm z_{1-\frac{\alpha}{2}}\left(\frac{1}{\sum_{j=1}^{n_1} y_{1j}}+\frac{1}{\sum_{j=1}^{n_0} y_{0j}}\right)^{1/2}\right]$$

$$= (0{,}829,\ 2{,}056)$$ contient $\underline{1}$.

Exercice 2

(a) Soit $\pi(x_i) = \dfrac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}$. La vraisemblance est donnée :

$$\mathcal{L} = \prod_{i=1}^{n} \pi_i^{y_i}(1-\pi_i)^{1-y_i}$$

donc : $$\ln \mathcal{L} = \sum_{i=1}^{n} y_i \ln(\pi_i) + (1-y_i)\ln(1-\pi_i).$$

Formule de dérivée ~~fon~~ en chaîne (fonctions composées) :

$$\frac{\partial \ln \mathcal{L}}{\partial \alpha} = \sum_{i=1}^{n} \frac{\partial \ln \mathcal{L}}{\partial \pi_i}\cdot\frac{\partial \pi_i}{\partial \alpha} \quad \text{et} \quad \frac{\partial \ln \mathcal{L}}{\partial \beta} = \sum_{i=1}^{n} \frac{\partial \ln \mathcal{L}}{\partial \pi_i}\cdot\frac{\partial \pi_i}{\partial \beta}$$

Nous avons : $$\frac{\partial \pi_i}{\partial \alpha} = \frac{\partial\left[\dfrac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}\right]}{\partial \alpha} = \frac{e^{\alpha+\beta x_i}}{(1+e^{\alpha+\beta x_i})^2} = \pi_i(1-\pi_i)$$

$$\frac{\partial \pi_i}{\partial \beta} = \frac{\partial\left[\dfrac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}\right]}{\partial \beta} = \frac{x_i\, e^{\alpha+\beta x_i}}{(1+e^{\alpha+\beta x_i})^2} = x_i\pi_i(1-\pi_i).$$

et : $$\frac{\partial \ln \mathcal{L}}{\partial \alpha} = \sum_{i=1}^{n} \frac{\partial \ln \mathcal{L}}{\partial \pi_i}\frac{\partial \pi_i}{\partial \alpha} = \sum_{i=1}^{n}\left[\frac{y_i}{\pi_i} - \frac{1-y_i}{1-\pi_i}\right]\pi_i(1-\pi_i)$$

$$= \sum_{i=1}^{n}\left[y_i(1-\pi_i) - (1-y_i)\pi_i\right] = \sum_{i=1}^{n}\left[y_i - \cancel{y_i\pi_i} - \pi_i + \cancel{y_i\pi_i}\right]$$

$$= \sum_{i=1}^{n}\left[y_i - \pi_i\right] = 0$$

$$\Rightarrow \sum_{i=1}^{n} y_i = \sum_{i=1}^{n}\pi_i = \sum_{i=1}^{n}\frac{e^{\alpha+\beta x_i}}{(1+e^{\alpha+\beta x_i})} =$$

$$= \sum_{i=1}^{n_0}\frac{e^{\alpha}}{1+e^{\alpha}} + \sum_{i=n_0+1}^{n}\frac{e^{\alpha+\beta}}{1+e^{\alpha+\beta}}$$

$$\Rightarrow \sum_{i=1}^{n} y_i = n_0 \frac{e^{\alpha}}{(1+e^{\alpha})} + n_1 \frac{e^{\alpha+\beta}}{1+e^{\alpha+\beta}}.$$

De manière similaire :

$$\frac{\partial \ln \mathcal{L}}{\partial \beta} = \sum_{i=1}^{n} \frac{\partial \ln \mathcal{L}}{\partial \pi_i} \frac{\partial \pi_i}{\partial \beta} = \sum_{i=1}^{n} \left[\frac{y_i}{\pi_i} - \frac{1-y_i}{1-\pi_i} \right] x_i \pi_i (1-\pi_i)$$

$$= \sum_{i=1}^{n} x_i (y_i - \pi_i) = \sum_{i=n_0+1}^{n} (y_i - \pi_i) = 0 ;$$

$$\Rightarrow \sum_{i=n_0+1}^{n} y_i = \sum_{i=n_0+1}^{n} \pi_i = \sum_{i=n_0+1}^{n} \frac{e^{\alpha+\beta}}{1+e^{\alpha+\beta}} = n_1 \frac{e^{\alpha+\beta}}{(1+e^{\alpha+\beta})}$$

$$\sum_{i=1}^{n} y_i - \sum_{i=n_0+1}^{n} y_i = \frac{n_0 e^{\alpha}}{1+e^{\alpha}} \Rightarrow \sum_{i=1}^{n_0} y_i = n_0 \frac{e^{\alpha}}{1+e^{\alpha}}.$$

$$\Rightarrow \hat{\alpha} = \ln \left(\frac{p_0}{1-p_0} \right) \quad \text{où} \quad p_0 = \frac{1}{n_0} \sum_{i=1}^{n_0} y_i$$

Il s'en suit que : $\displaystyle \sum_{i=n_0+1}^{n} y_i = n_0 \frac{e^{\hat{\alpha}+\hat{\beta}}}{1+e^{\hat{\alpha}+\hat{\beta}}} \Rightarrow \hat{\beta} = \ln \left[\frac{p_1}{1-p_1} \right] - \hat{\alpha}$

$$\hat{\beta} = \ln \left[\frac{p_1}{1-p_1} \right] - \ln \left[\frac{p_0}{1-p_0} \right] = \ln \left[\frac{p_1/(1-p_1)}{p_0/(1-p_0)} \right]$$

$$p_1 = \frac{1}{n_1} \sum_{i=n_0+1}^{n} y_i .$$

(b) Matrice de variance-covariance quand n est grand :

$$-\frac{\partial^2 \ln \mathcal{L}}{\partial \alpha^2} = \frac{-\partial \sum_{i=1}^{n}(y_i - \pi_i)}{\partial \pi_i} \frac{\partial \pi_i}{\partial \alpha}.$$

$$= \sum_{i=1}^{n}(1-\pi_i)\pi_i = \sum_{i=1}^{n} \frac{e^{\alpha+\beta x_i}}{(1+e^{\alpha+\beta x_i})^2}$$

$$= \sum_{i=1}^{n_0} \frac{e^{\alpha}}{(1+e^{\alpha})^2} + \sum_{i=n_0+1}^{n} \frac{e^{\alpha+\beta}}{(1+e^{\alpha+\beta})^2}$$

$$= n_0 \pi_0(1-\pi_0) + n_1 \pi_1(1-\pi_1)$$

$$\pi_0 = \frac{e^{\alpha}}{1+e^{\alpha}} \quad \text{et} \quad \pi_1 = \frac{e^{\alpha+\beta}}{1+e^{\alpha+\beta}}.$$

Aussi :

$$-\frac{\partial^2 \ln \mathcal{L}}{\partial \beta^2} = \frac{-\partial \sum_{i=1}^{n} x_i(y_i - \pi_i)}{\partial \pi_i} \frac{\partial \pi_i}{\partial \beta}$$

$$= \sum_{i=1}^{n} x_i^2 \pi_i (1-\pi_i)$$

$$= \sum_{i=n_0+1}^{n} \frac{e^{\alpha+\beta}}{(1+e^{\alpha+\beta})^2} = n_1 \pi_1 (1-\pi_1)$$

Finalement :

$$-\frac{\partial^2 \ln \mathcal{L}}{\partial \alpha \partial \beta} = -\frac{\partial \sum_{i=1}^{n}(y_i - \pi_i)}{\partial \pi_i} \frac{\partial \pi_i}{\partial \beta} = \sum_{i=1}^{n} x_i \pi_i (1-\pi_i)$$

$$= n_1 \pi_1 (1-\pi_1) = -\frac{\partial^2 \ln \mathcal{L}}{\partial \beta \partial \alpha}.$$

Avec $y = (y_1, y_2, \ldots, y_n)$, nous avons

$$I(y; \alpha, \beta) = \begin{bmatrix} n_0 \pi_0(1-\pi_0) + n_0 \pi_1(1-\pi_1) & n_1 \pi_1(1-\pi_1) \\ n_1 \pi_1(1-\pi_1) & n_1 \pi_1(1-\pi_1) \end{bmatrix}$$

$$I^{-1}(\alpha,\beta) = \begin{pmatrix} \left[n_0 \dfrac{e^{\alpha}}{(1+e^{\alpha})^2}\right]^{-1} & -\left[n_0 \dfrac{e^{\alpha}}{(1+e^{\alpha})^2}\right]^{-1} \\ -\left[n_0 \dfrac{e^{\alpha}}{(1+e^{\alpha})^2}\right]^{-1} & \left[n_0 \dfrac{e^{\alpha}}{(1+e^{\alpha})^2}\right]^{-1} + \left[n_1 \dfrac{e^{\alpha+\beta}}{(1+e^{\alpha+\beta})^2}\right]^{-1} \end{pmatrix}$$

(c) le IC à 95 % estimé :

$$\hat{\alpha} \pm 1{,}96\sqrt{\left[n_0 \frac{e^{\hat{\alpha}}}{(1+e^{\hat{\alpha}})^2}\right]^{-1}}$$

et le IC à 95 % pour β :

$$\hat{\beta} \pm 1{,}96\sqrt{\left[n_0 \frac{e^{\hat{\alpha}}}{(1+e^{\hat{\alpha}})^2}\right]^{-1} + \left[n_1 \frac{e^{\hat{\alpha}+\hat{\beta}}}{(1+e^{\hat{\alpha}+\hat{\beta}})^2}\right]^{-1}}$$

à partir des données $\hat{\alpha} = -1.52$ et $\hat{\beta} = 0{,}47$

$IC(\alpha) = [-2{,}03 , -1{,}01]$ et $[-0.21 , 1{,}15]$.

Exercice ③

Soit $\alpha, \beta > 0$, $\theta = (\alpha, \beta)$ et on définit la densité de probabilité

$$f_\theta(x) = \alpha \beta^{-\alpha} x^{\alpha - 1} \mathbb{1}_{[0,\beta]}(x)$$

On pose $U = -\alpha \log\left(\frac{X}{\beta}\right)$ la loi de $U \sim \Gamma(1,1) \simeq \mathcal{E}(1)$

$X_1, \dots, X_n$ iid $\sim f_\theta$.

$$c = \frac{[\mathbb{E}(X)]^2}{Var(X)} = \frac{\cancel{\alpha^2} \cancel{\beta^2}}{\cancel{(\alpha+1)^2}} \times \frac{\cancel{(\alpha+1)^2}(\alpha+2)}{\cancel{\alpha} \cancel{\beta^2}}$$

$$c = \alpha(\alpha+2)$$

$\Rightarrow \alpha^2 + 2\alpha - c = 0$ <u>solution</u> : $\Delta = b^2 - 4ac$

$\Rightarrow \Delta = 4 + 4c$

$= 4(1+c) > 0$

deux solutions : $\alpha_1 = \frac{-b - \sqrt{\Delta}}{2a}$ $\alpha_2 = \frac{-b+\sqrt{\Delta}}{2a}$

$\alpha_1 = \frac{-\cancel{2} - \cancel{2}\sqrt{1+c}}{\cancel{2}}$ $\alpha_2 = \frac{-\cancel{2} + \cancel{2}\sqrt{1+c}}{\cancel{2}}$

$= \sqrt{1+c} - 1.$

une seule solution : $\hat{\alpha} = \sqrt{1+c} - 1$

$$\hat{\alpha} = \sqrt{1 + \frac{(\bar{X}_n)^2}{\hat{\sigma}_n^2}} - 1.$$

②

3) β connu, MLE de α :

vraisemblance : $L_n(\alpha) = \prod_{i=1}^{n} \alpha \beta^{-\alpha} x_i^{\alpha-1} = \alpha^n \beta^{-n\alpha} \prod_{i=1}^{n} x_i^{-1} e^{\alpha \sum_{i=1}^{n} \log x_i}$

$$\boxed{x_i^{\alpha} = e^{\log x_i^{\alpha}} = e^{\alpha \log x_i}}$$

Log-vraisemblance :

$$l_n(\alpha) = n \log \alpha - n\alpha \log \beta - \sum_{i=1}^{n} \log x_i + \alpha \sum_{i=1}^{n} \log x_i$$

$$\frac{\partial l_n(\alpha)}{\partial \alpha} = \frac{n}{\alpha} - n \log \beta + \sum_{i=1}^{n} \log x_i$$

$$= \frac{n}{\alpha} + \sum_{i=1}^{n} \log \frac{x_i}{\beta} = 0 \iff \frac{n}{\alpha} = -\sum_{i=1}^{n} \log \frac{x_i}{\beta}$$

$$\iff \alpha = -\frac{n}{\sum_{i=1}^{n} \log \frac{x_i}{\beta}}$$

$$\Rightarrow \hat{\alpha}_n = \frac{\alpha}{\bar{U}_n}.$$

On sait que $\mathbb{E}[\bar{U}_n] = 1$ et ~~donc~~

$$\underbrace{\mathbb{E}\left[\frac{1}{\bar{U}_n}\right] > \frac{1}{\mathbb{E}[\bar{U}_n]}}_{\text{Jensen}} \quad \underline{\text{donc}} \quad \mathbb{E}[\hat{\alpha}_n] > \alpha.$$

4) $\tilde{\alpha}_n = \frac{n-1}{n} \hat{\alpha}_n$ montrons que $\tilde{\alpha}_n$ est sans biais

$\underline{\text{dm}}$ $\tilde{\alpha}_n = \frac{n-1}{\sum_{i=1}^{n} U_i} \alpha$ or $\sum_{i=1}^{n} U_i \sim \Gamma(n,1)$

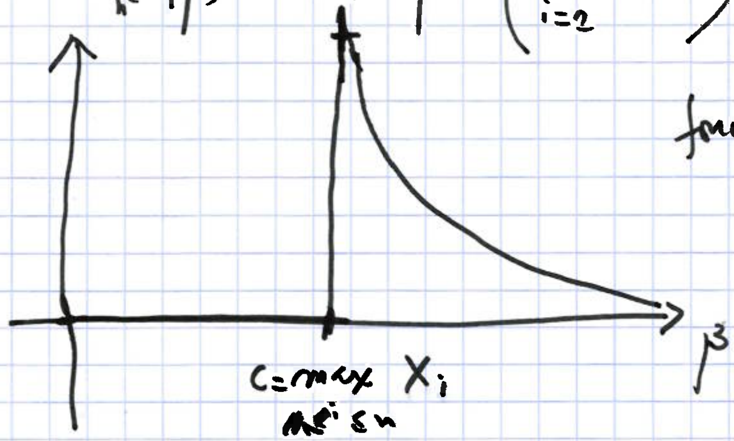
③ $$\mathbb{E}\left[\tilde{\alpha}_n\right] = (n-1)\,\alpha \int_0^{+\infty} \frac{1}{x}\,\frac{x^{n-1}}{(n-1)!}\,e^{-x}\,dx$$

$$= \frac{\alpha}{(n-2)!}\underbrace{\int_0^{+\infty} x^{n-2}\,e^{-x}\,dx}_{\Gamma(n-1)\,=\,(n-2)!}$$

$$= \alpha.$$

5) β inconnu :

$$L_n(\alpha,\beta) = \alpha^n\,\beta^{-n\alpha}\left(\prod_{i=1}^{n} x_i^{\alpha-1}\right)\mathbb{1}\left\{\max_{1\le i\le n} X_i \le \beta\right\}$$

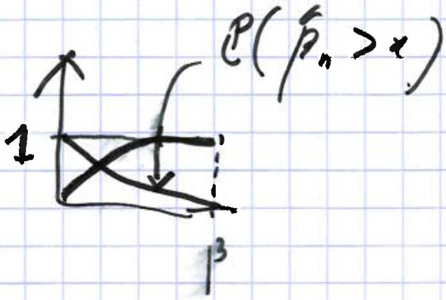


fonction nulle sur $\beta \in \left[0, \max_{1\le i\le n} X_i\right[$.

décroissante sur $\left[\max_{1\le i\le n} X_i, +\infty\right[$

$$\mathbb{P}\left(\hat{\beta}_n < \beta\right) = 1 \Rightarrow \mathbb{E}\left[\hat{\beta}_n\right] < \beta.$$

$$\mathbb{E}\left(\hat{\beta}_n\right) = \int_0^{\beta} \mathbb{P}\left(\hat{\beta}_n > x\right) < \beta.$$



①

Exercice ④

Question (1) : La vraisemblance $\mathcal{L}(t_1, t_2, \dots, t_n) \equiv \mathcal{L}$ — notation à adopter par la suite.

$$\mathcal{L} = \prod_{i=1}^{n} f_T(t_i;\theta) = \prod_{i=1}^{n}\left[\theta e^{-\theta t_i}\right] = \theta^n e^{-\theta \sum_{i=1}^{n} t_i}.$$

Donc,

$$\ln \mathcal{L} = n \ln\theta - \theta \sum_{i=1}^{n} t_i, \quad \frac{\partial \ln \mathcal{L}}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} t_i$$

et,

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2} = \frac{-n}{\theta^2}.$$

La variance asymptotique (quand n est grand) de $\hat{\theta}_n$ est donnée par

$$Var(\hat{\theta}_n) = \left[-\mathbb{E}\left(\frac{\partial^2 \ln \mathcal{L}}{\partial \theta^2}\right)\right]^{-1} = \frac{\theta^2}{n}.$$

Question (2) : Nous avons,

$$\underbrace{\mathbb{P}(T_i > t^*)}_{\text{la fonction de survie en } t} = \int_{t^*}^{+\infty} \theta e^{-\theta t}\, dt = \left[-\theta e^{-\theta t}\right]_{t^*}^{+\infty} = e^{-\theta t^*}.$$

Ainsi, la vraisemblance $\mathcal{L}^*(y_1, y_2, \dots, y_n) \equiv \mathcal{L}^*$ est donnée par

$$\mathcal{L}^* = \prod_{i=1}^{n}\left\{\left(e^{-\theta t^*}\right)^{y_i}\left(1 - e^{-\theta t^*}\right)^{1-y_i}\right\}$$

car Y_i est Bernoulli de proba de succès $e^{-\theta t^*}$.

$$\mathcal{L}^* = e^{-\theta t^* \sum_{i=1}^{n} y_i}\left(1 - e^{-\theta t^*}\right)^{n - \sum_{i=1}^{n} y_i}$$

donc,

$$\ln \mathcal{L}^* = -\theta t^* n\bar{y}_n + n(1-\bar{y}_n)\ln\left(1 - e^{-\theta t^*}\right) \quad \text{où } \bar{y}_n = \frac{1}{n}\sum_{i} y_i$$

$$\frac{\partial \ln \mathcal{L}^*}{\partial \theta} = -t^* n \bar{y}_n + n(1-\bar{y}_n) \times \frac{t^* e^{-\theta t^*}}{\left(1 - e^{-\theta t^*}\right)}$$

(2)

$$\frac{\partial \ln \mathcal{L}^*}{\partial \theta} = 0 \Rightarrow n(1-\bar{y}_n)\, t^* e^{-\theta t^*} = n t^* \bar{y}_n (1 - e^{-\theta t^*})$$

$$\Rightarrow (1-\bar{y}_n) e^{-\theta t^*} = \bar{y}_n (1 - e^{-\theta t^*})$$

$$\Rightarrow e^{-\theta t^*} = \bar{y}_n \Rightarrow \hat{\theta}_n^* = \frac{-\ln \bar{Y}_n}{t^*}$$

$$= \frac{1}{t^*} \ln\left(\frac{1}{\bar{Y}_n}\right).$$

Question (3) :

$$\frac{\partial^2 \ln \mathcal{L}^*}{\partial \theta^2} = n t^* (1-\bar{y}_n) \left[\frac{-t^* e^{-\theta t^*}(1 - e^{-\theta t^*}) - e^{-\theta t^*}(t^* e^{-\theta t^*})}{(1 - e^{-\theta t^*})^2}\right]$$

$$= \frac{-n t^* (1 - \bar{y}_n)}{(1 - e^{-\theta t^*})^2} t^* e^{-\theta t^*}.$$

On déduit :

$$-\mathbb{E}\left[\frac{\partial^2 \ln \mathcal{L}^*}{\partial \theta^2}\right] = \frac{n (t^*)^2 e^{-\theta t^*}\, \mathbb{E}[1 - \bar{Y}_n]}{(1 - e^{-\theta t^*})^2}$$

$$= \frac{n (t^*)^2 e^{-\theta t^*}}{(1 - e^{-\theta t^*})^2}(1 - e^{-\theta t^*})$$

$$= \frac{n (t^*)^2}{(e^{\theta t^*} - 1)}.$$

$\hat{\theta}_n$ est préférable car les $Y_1, \dots, Y_n$ perdent de l'information par rapport aux $T_1, \dots, T_n$ si les temps $T_1, \dots, T_n$ sont mesurées sans erreur.

La variance asymptotique de $\hat{\theta}_n^*$ est égale à $\dfrac{(e^{\theta t^*} - 1)}{n (t^*)^2}$.

Si $t^* \geq \mathbb{E}[T] = \frac{1}{\theta}$, nous avons :

$$\frac{\operatorname{Var}(\hat{\theta}_n)}{\operatorname{Var}(\hat{\theta}_n^*)} = \frac{\theta^2/n}{(e^{\theta t^*} - 1)/n(t^*)^2} = \frac{\theta^2 (t^*)^2}{(e^{\theta t^*} - 1)} < 1,$$

③

Exercice ⑤ :

Question (1) : On note que :

$$\mu_r = \mathbb{E}[Y^r] = \int_\gamma^{+\infty} y^r \,\theta \gamma^\theta y^{-(\theta+1)}\, dy$$

$$= \theta\gamma^\theta \left[\frac{y^{r-\theta}}{(r-\theta)} \right]_\gamma^{+\infty} = \frac{\theta\gamma^r}{(\theta-r)} \text{ , pour } \theta > r.$$

La méthode des moments repose sur les deux équations suivantes :

$$\begin{cases} \hat{\mu}_1 = \bar{y}_n = \dfrac{\theta\gamma}{\theta-1} = \mathbb{E}[Y] \quad \text{------ ①} \\ \hat{\mu}_2 = \dfrac{1}{n}\sum_{i=1}^n y_i^2 = \mathbb{E}[Y^2] = \dfrac{\theta\gamma^2}{(\theta-2)} \quad \text{----- ②} \end{cases}$$

Les deux équations précédentes donnent :

$$\frac{\hat{\mu}_2}{\bar{y}_n^2} = \frac{\theta\gamma^2/(\theta-2)}{\theta^2\gamma^2/(\theta-1)^2} = \frac{(\theta-1)^2}{\theta(\theta-2)}$$

Ainsi :

$$\frac{(\theta-1)^2}{\theta(\theta-2)} - 1 = \frac{1}{\theta(\theta-2)} = \frac{\hat{\mu}_2}{\bar{y}_n^2} - 1 = \frac{(\hat{\mu}_2 - \bar{y}_n^2)}{\bar{y}_n^2}$$

$$= \frac{\frac{1}{n}\sum_{i=1}^n (y_i - \bar{y}_n)^2}{\bar{y}_n^2} = \left(\frac{n-1}{n}\right)\frac{s^2}{\bar{y}_n^2}.$$

donc :

$$\theta(\theta-2) = \left(\frac{n}{n-1}\right)\frac{\bar{y}_n^2}{s^2} = \left(\frac{50}{49}\right)\left(\frac{900}{10}\right) = 91.8367$$

Les solutions de l'équation du second degré : $\theta^2 - 2\theta - 91.8367 = 0$

④ sont données par :

$$\frac{2 \pm \sqrt{(-2)^2 + 4(91.8367)}}{2}, \text{ ou } \begin{cases} -8.6352 \\ 10.6352 \end{cases}$$

Comme $\theta > 2$, on garde la solution positive $\hat{\theta}_{mm} = 10.6352$.

Finalement

$$\hat{\gamma}_{mm} = \frac{(\hat{\theta}_{mm} - 1)}{\hat{\theta}_{mm}} \bar{y}_n = \left(\frac{9.6352}{10.6352}\right)(30) = 27.1793.$$

donc : $\hat{\gamma}_{mm} = 27.1793$.

$$f_{Y_{(n)}}(y;\gamma,\theta) = n\left[1 - F_Y(y;\gamma,\theta)\right]^{n-1} f_Y(y;\gamma,\theta)$$

$$= n\left[\left(\frac{\gamma}{y}\right)^{\theta}\right]^{n-1} \theta\gamma^{\theta} y^{-(\theta+1)}$$

$$= n\theta\gamma^{n\theta} y^{-(n\theta+1)}, \quad 0 < \gamma < y < +\infty.$$

À l'aide de cette densité on peut calculer :

$$\mathbb{E}\left[Y_{(n)}^r\right] = \int_{\gamma}^{+\infty} y^r n\theta\gamma^{n\theta} y^{-(n\theta+1)} dy = \frac{n\theta\gamma^r}{(n\theta - r)}, \quad n\theta > r.$$

donc : $\mathbb{E}\left[Y_{(n)}\right] = \frac{n\theta\gamma}{(n\theta - 1)}$.

$$\lim_{n\to+\infty} \mathbb{E}\left[Y_{(n)}\right] = \lim_{n\to+\infty} \frac{\theta\gamma}{(\theta - \frac{1}{n})} = \frac{\theta}{\theta}\gamma = \gamma.$$

$$\operatorname{Var}(Y_{(n)}) = \frac{n\theta\gamma^2}{(n\theta - 2)} - \left[\frac{n\theta\gamma}{(n\theta - 1)}\right]^2 = n\theta\gamma^2\left[\frac{1}{(n\theta - 2)} - \frac{n\theta}{(n\theta - 1)^2}\right]$$

$$= \frac{n\theta\gamma^2}{(n\theta - 1)^2(n\theta - 2)}.$$

$$\lim_{n\to+\infty} \operatorname{Var}(Y_{(1)}) = 0$$

donc $Y_{(1)}$ converge en proba vers γ.

⑤

Question (3) : Soit $c = (1-\alpha)^{\frac{1}{n\theta}}$; nous avons $U = c\, Y_{(1)}$

$$= (1-\alpha)^{\frac{1}{3n}}\, Y_{(1)}.$$

Lorsque $n = 5$, $\alpha = 0.1$ et $y_{(1)} = 20$, la valeur calculée de U, $u = (1-0.1)^{1/15}(20) = 19.860..$

Le IC à 90% pour γ est donné par $[0, 19.860]$.